

Blow-up in reaction-diffusion systems under Robin boundary conditions

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Abstract

In this paper we apply the differential inequality technique of Payne *et. al* [10] to show that a reaction-diffusion system admits blow-up solutions, and to determine an upper bound for the blow-up time. For a particular nonlinearity, a lower bound on the blow-up time, when blow-up does occur, is also given.

1 Introduction

Since the pioneering work of Fujita ([4]) on the blow-up of solutions of nonlinear diffusion equations, there has been considerable interest in the study of such solutions for nonlinear parabolic equations. Blow-up phenomena can be observed in nature and is important in various disciplines such as biology, chemistry and physics. More detailed results related to the blow-up of solutions can be seen, for example, in the monograph [12, 13, 15], the surveys [1, 3, 5, 14] and bibliographies cited therein.

Recently, Payne and Schaefer ([6, 7, 8, 9, 10, 11]) have applied the *energy method* to derive a differential inequality for certain integrals corresponding to the blow-up solutions of parabolic equations. By means of this method, they find lower bounds on the blow-up time in certain nonlinear parabolic problems. In the present paper, we find that the above-mentioned method can be applied to *reaction-diffusion systems* to obtain generalizations of results in [10]. More precisely, it can be shown that Theorem 2.1 and Theorem 3.1 in [10] can be extended to Theorem 1.1 and Theorem 3.1, respectively in this paper.

Compared with cases of single equations, little work appears to have been devoted to systems. Motivated by Theorem 2.1 in [10], we show that the result in Theorem 2.1 can be extended to the *gradient system* case. To be specific, we consider the following

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initial-boundary value problem for the reaction-diffusion system under a Robin boundary condition, i.e.

$$\begin{cases} u_t = \Delta u + f_1(u, v), & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta v + f_2(u, v), & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} + \gamma_1 u = 0, \quad \frac{\partial v}{\partial \nu} + \gamma_2 v = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = g_1(x), \quad v(x, 0) = g_2(x) & \text{in } \bar{\Omega}, \end{cases} \quad (1.1)$$

where $(u, v) = (u(x, t), v(x, t))$; $\Omega \in \mathbb{R}^N$, $N \geq 2$ is a bounded domain with smooth $\partial\Omega$; γ_1 and γ_2 are positive constants; ν is the unit outward normal on $\partial\Omega$; and $g_1(x)$ and $g_2(x)$ are nonnegative functions which do not completely vanish. Suppose that the nonlinearity $f_1(u, v)$ and $f_2(u, v)$ in (1.1) satisfy

$$(H1) \quad u f_1(u, v) + v f_2(u, v) \geq 2(1 + \alpha) F(u, v),$$

where $F(u, v)$ is a solution to $\partial_u F(u, v) = f_1(u, v)$, $\partial_v F(u, v) = f_2(u, v)$, and $\alpha > 0$ is a constant. In addition to the non-negativity of the initial conditions $g_1(x)$ and $g_2(x)$, we impose the additional conditions:

$$(H2) \quad 2 \int_{\Omega} F(g_1(x), g_2(x)) dx \geq \gamma_1 \int_{\partial\Omega} g_1^2 ds + \int_{\Omega} |\nabla g_1|^2 dx;$$

$$(H3) \quad 2 \int_{\Omega} F(g_1(x), g_2(x)) dx \geq \gamma_2 \int_{\partial\Omega} g_2^2 ds + \int_{\Omega} |\nabla g_2|^2 dx.$$

Under these conditions we have:

Theorem 1.1. *Suppose that $(u(x, t), v(x, t))$ is a pair of solution to (1.1). If $(H1) \sim (H3)$ are satisfied, then at least one of $u(x, t)$ and $v(x, t)$ blows up in finite time t^* , where t^* is bounded above by (2.10).*

The physical meaning of the Robin boundary conditions can be explained as follows. Suppose that u and v represent temperature, and are governed by the equations in the problem (1.1). Then the Robin boundary conditions mean that the heat flux $\frac{\partial u}{\partial \nu}$ and $\frac{\partial v}{\partial \nu}$ on the boundary of Ω are proportional to the temperature u and v on the boundary of Ω , respectively. Since γ_1 and γ_2 are positive constants, it follows that the larger the heat flux is, the smaller the temperature is. We note that, from the biological point of view, the temperature and the heat flux can be substituted respectively to population density and population flux.

In other words, the larger the population flux is, the smaller the population density is. As a consequence, when the population flux on $\partial\Omega$ is large, the population density on $\partial\Omega$ is small. The low density of u and v on $\partial\Omega$ then may result in the blow-up of u or v since the large flux flows into Ω but on the boundary of Ω , the density of u and v are restricted to be small. Therefore, u and v are may be forced to aggregate together so that blow-up occurs.

The remainder of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.1. This theorem asserts that for certain initial conditions, the solutions of

(1.1) blow up in finite time, with an upper bound given by (2.10). Section 3 is devoted to determining a lower bound on the blow-up time when blow-up does occur. In addition, Theorem 3.2 provides a cooperative system considered in [2] a lower bounded for blow-up time when blow-up does occur. Finally, we conclude the present paper with some remarks in Section 4.

2 Proof of Theorem 1.1

In this section, Theorem 1.1 is proven.

Proof. First of all, we define

$$E(t) = \int_{\Omega} (u^2 + v^2) dx. \quad (2.1)$$

By means of integration by parts and (H1), we arrive at

$$\begin{aligned} E'(t) &= 2 \int_{\Omega} u (\Delta u + f_1(u, v)) + v (\Delta v + f_2(u, v)) dx \\ &= -2 \gamma_1 \int_{\partial\Omega} u^2 ds - 2 \int_{\Omega} |\nabla u|^2 dx + 2 \int_{\Omega} u f_1(u, v) dx \\ &\quad - 2 \gamma_2 \int_{\partial\Omega} v^2 ds - 2 \int_{\Omega} |\nabla v|^2 dx + 2 \int_{\Omega} v f_2(u, v) dx \\ &\geq -2(1 + \alpha) (\gamma_1 \int_{\partial\Omega} u^2 ds + \int_{\Omega} |\nabla u|^2 dx) - 2(1 + \alpha) (\gamma_2 \int_{\partial\Omega} v^2 ds + \int_{\Omega} |\nabla v|^2 dx) \\ &\quad + 4(1 + \alpha) \int_{\Omega} F(u, v) dx. \end{aligned} \quad (2.2)$$

Letting

$$\begin{aligned} J(t) &= -2(1 + \alpha) (\gamma_1 \int_{\partial\Omega} u^2 ds + \int_{\Omega} |\nabla u|^2 dx) - 2(1 + \alpha) (\gamma_2 \int_{\partial\Omega} v^2 ds + \int_{\Omega} |\nabla v|^2 dx) \\ &\quad + 4(1 + \alpha) \int_{\Omega} F(u, v) dx, \end{aligned} \quad (2.3)$$

we calculate the derivative of $J(t)$ to obtain

$$\begin{aligned}
J'(t) &= -4(1+\alpha)(\gamma_1 \int_{\partial\Omega} u u_t ds + \int_{\Omega} \nabla u \cdot \nabla u_t dx - \int_{\Omega} f_1(u, v) u_t dx) \\
&\quad -4(1+\alpha)(\gamma_2 \int_{\partial\Omega} v v_t ds + \int_{\Omega} \nabla v \cdot \nabla v_t dx - \int_{\Omega} f_2(u, v) v_t dx) \\
&= -4(1+\alpha)(\gamma_1 \int_{\partial\Omega} u u_t ds + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u_t ds - \int_{\Omega} u_t (\Delta u + f_1(u, v)) \\
&\quad -4(1+\alpha)(\gamma_2 \int_{\partial\Omega} v v_t ds + \int_{\partial\Omega} \frac{\partial v}{\partial \nu} v_t ds - \int_{\Omega} v_t (\Delta v + f_2(u, v)) \\
&\geq 4(1+\alpha) \int_{\Omega} (u_t^2 + v_t^2) dx,
\end{aligned} \tag{2.4}$$

by virtue of integration by parts and the Robin boundary conditions in (1.1). Due to (H2) and (H3), we have $J(0) \geq 0$. Then $J(0) \geq 0$ and $J'(t) \geq 0$, for $t \geq 0$, yield $J(t) \geq 0$ for $t \geq 0$. Because

$$E'(t) = 2 \int_{\Omega} (u u_t + v v_t) dx, \tag{2.5}$$

we have by applying Cauchy-Schwartz inequality

$$\begin{aligned}
(E'(t))^2 &= 4 \left(\left(\int_{\Omega} u u_t dx \right)^2 + \left(\int_{\Omega} v v_t dx \right)^2 + 2 \left(\int_{\Omega} u u_t dx \right) \left(\int_{\Omega} v v_t dx \right) \right) \\
&\leq 4 \left(\int_{\Omega} u^2 dx \int_{\Omega} u_t^2 dx + \int_{\Omega} v^2 dx \int_{\Omega} v_t^2 dx \right. \\
&\quad \left. + 2 \left(\int_{\Omega} u^2 dx \int_{\Omega} u_t^2 dx \int_{\Omega} v^2 dx \int_{\Omega} v_t^2 dx \right)^{\frac{1}{2}} \right) \\
&\leq 4 \int_{\Omega} (u^2 + v^2) dx \int_{\Omega} (u_t^2 + v_t^2) dx \\
&\leq \frac{1}{1+\alpha} E(t) J'(t).
\end{aligned} \tag{2.6}$$

The last inequality holds since $\sqrt{ab} \leq \frac{a+b}{2}$, for $a, b \geq 0$. From the definition of $J(t)$, $E'(t) \geq J(t)$, and consequently we can replace $E'(t)$ by $J(t)$ in (2.6) to obtain

$$J(t) E'(t) \leq \frac{1}{1+\alpha} E(t) J'(t) \tag{2.7}$$

or

$$(1+\alpha) \frac{E'(t)}{E(t)} \leq \frac{J'(t)}{J(t)}. \tag{2.8}$$

Following the same arguments in [10], we obtain that $\varphi(t)$ satisfies the inequality

$$\frac{1}{(E(t))^\alpha} \leq \frac{1}{(E(0))^\alpha} - \alpha M t, \tag{2.9}$$

where $M = \frac{J(0)}{(E(0))^{1+\alpha}}$. Since the last inequality cannot be true for all $t \geq 0$, we infer that at least one of $u(x, t)$ and $v(x, t)$ blows up in finite time, t^* , where t^* is bounded above by

$$t^* \leq \frac{1}{\alpha M (E(0))^\alpha}. \quad (2.10)$$

□

We note that Theorem 1.1 remains true if the Robin boundary conditions are replaced by Neumann boundary conditions (i.e. $\gamma_1 = \gamma_2 = 0$). To illustrate the results in Theorem 1.1, we give an example of the nonlinearity in (1.1). Take $F(u, v) = u^2 v^3$, then $f_1 = \partial_u F(u, v) = 2 u v^3$, $f_2 = \partial_v F(u, v) = 3 u^2 v^2$,

$$u f_1(u, v) + v f_2(u, v) = 5 u^2 v^3, \quad (2.11)$$

and

$$2(1 + \alpha) F(u, v) = 2(1 + \alpha) u^2 v^3. \quad (2.12)$$

Clearly, (H1) is fulfilled when $0 < \alpha \leq \frac{3}{2}$.

Remark 2.1. Equivalently, (H1) can be rewritten as

$$u \partial_u F(u, v) + v \partial_v F(u, v) \geq 2(1 + \alpha) F(u, v). \quad (2.13)$$

For equality, that is

$$u \partial_u F(u, v) + v \partial_v F(u, v) = 2(1 + \alpha) F(u, v), \quad (2.14)$$

which can be solved by *the method of characteristics* to give

$$F(u, v) = c u^{2(1+\alpha)} h\left(\frac{v}{u}\right), \quad (2.15)$$

where $h = h(w)$ is an arbitrary smooth function and c is an arbitrary constant. This solution is useful in looking for nonlinearities f_1 and f_2 in (1.1) which satisfy (H1).

3 Lower bound for blow-up time

In this section, a lower bound on the blow-up time is obtained when blow-up does occur. In particular, (2.16) in [8] plays an essential role in proving the following

Theorem 3.1. *Let $(u(x, t), v(x, t))$ be a pair of nonnegative solutions to (1.1) and at least one of $u(x, t)$ and $v(x, t)$ blow up in finite time, $t = t^*$. Suppose that (A1) \sim (A3) below are also satisfied:*

(A1) $\Omega \in \mathbb{R}^3$ is a bounded smooth convex domain;

(A2) $f_1(u, v) \leq k_1 u^{p+1}$ for $k_1, u, v > 0$ and $p \geq 1$;

(A3) $f_2(u, v) \leq k_2 v^{p+1}$ for $k_2, u, v > 0$ and $p \geq 1$.

Then t^* is bounded below by (3.12) .

Proof. First let us define the auxiliary function

$$\mathcal{E}(t) = \int_{\Omega} (u^{2p} + v^{2p}) dx. \quad (3.1)$$

On applying Green's first identity and the equality $u^{2p-2} |\nabla u|^2 = p^{-2} |\nabla u^p|^2$, we arrive at

$$\begin{aligned} \mathcal{E}'(t) &= 2p \int_{\Omega} u^{2p-1} (\Delta u + f_1(u, v)) + v^{2p-1} (\Delta v + f_2(u, v)) dx \\ &= -2p \gamma_1 \int_{\partial\Omega} u^{2p} ds - 2p(2p-1) \int_{\Omega} u^{2p-2} |\nabla u|^2 dx + 2p \int_{\Omega} u^{2p-1} f_1(u, v) dx \\ &\quad - 2p \gamma_2 \int_{\partial\Omega} v^{2p} ds - 2p(2p-1) \int_{\Omega} v^{2p-2} |\nabla v|^2 dx + 2p \int_{\Omega} v^{2p-1} f_2(u, v) dx \\ &\leq -2(2p-1)p^{-1} \int_{\Omega} |\nabla u^p|^2 dx + 2p \int_{\Omega} u^{2p-1} f_1(u, v) dx \\ &\quad - 2(2p-1)p^{-1} \int_{\Omega} |\nabla v^p|^2 dx + 2p \int_{\Omega} v^{2p-1} f_2(u, v) dx \\ &\leq -2(2p-1)p^{-1} \int_{\Omega} |\nabla u^p|^2 dx + 2p k_1 \int_{\Omega} u^{3p} dx \\ &\quad - 2(2p-1)p^{-1} \int_{\Omega} |\nabla v^p|^2 dx + 2p k_2 \int_{\Omega} v^{3p} dx \end{aligned} \quad (3.2)$$

since (A2) and (A3) hold. Now our strategy is to relate $\int_{\Omega} v^{3p} dx$ in terms of $\mathcal{E}(t)$ and $\int_{\Omega} |\nabla v^p|^2 dx$. To this end, the integral inequality (see (2.16) in [8]) is used:

$$\int_{\Omega} u^{3p} dx \leq \frac{1}{3^{\frac{3}{4}}} \left\{ \frac{3}{2\rho} \int_{\Omega} u^{2p} dx + \left(\frac{d}{\rho} + 1 \right) \left(\int_{\Omega} u^{2p} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u^p|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{3}{2}}, \quad (3.3)$$

where for some origin inside Ω ,

$$\rho = \min_{\partial\Omega} x_i \nu_i, \quad d^2 = \max_{\Omega} x_i x_i, \quad (3.4)$$

for ν_i the i -th component of the unit outward normal to $\partial\Omega$. Thus,

$$\begin{aligned}\mathcal{E}'(t) &\leq -2(2p-1)p^{-1} \int_{\Omega} |\nabla u^p|^2 dx - 2(2p-1)p^{-1} \int_{\Omega} |\nabla v^p|^2 dx \\ &\quad + \frac{2pk_1}{3^{\frac{3}{4}}} \left\{ \frac{3}{2\rho} \int_{\Omega} u^{2p} dx + \left(\frac{d}{\rho} + 1 \right) \left(\int_{\Omega} u^{2p} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla u^p|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{3}{2}} \\ &\quad + \frac{2pk_2}{3^{\frac{3}{4}}} \left\{ \frac{3}{2\rho} \int_{\Omega} v^{2p} dx + \left(\frac{d}{\rho} + 1 \right) \left(\int_{\Omega} v^{2p} dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |\nabla v^p|^2 dx \right)^{\frac{1}{2}} \right\}^{\frac{3}{2}}.\end{aligned}\tag{3.5}$$

By virtue of the elementary inequalities $(a+b)^{\frac{3}{2}} \leq 2^{\frac{1}{2}}(a^{\frac{3}{2}} + b^{\frac{3}{2}})$ and $a^{\frac{1}{4}}b^{\frac{3}{4}} \leq \frac{1}{4}a + \frac{3}{4}b$, we obtain

$$\begin{aligned}\mathcal{E}'(t) &\leq -2(2p-1)p^{-1} \int_{\Omega} |\nabla u^p|^2 dx - 2(2p-1)p^{-1} \int_{\Omega} |\nabla v^p|^2 dx \\ &\quad + \frac{2pk_1}{3^{\frac{3}{4}}} 2^{\frac{1}{2}} \left\{ \left(\frac{3}{2\rho} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{2p} dx \right)^{\frac{3}{2}} + \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{2p} dx \right)^{\frac{3}{4}} \left(\int_{\Omega} |\nabla u^p|^2 dx \right)^{\frac{3}{4}} \right\} \\ &\quad + \frac{2pk_2}{3^{\frac{3}{4}}} 2^{\frac{1}{2}} \left\{ \left(\frac{3}{2\rho} \right)^{\frac{3}{2}} \left(\int_{\Omega} v^{2p} dx \right)^{\frac{3}{2}} + \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \left(\int_{\Omega} v^{2p} dx \right)^{\frac{3}{4}} \left(\int_{\Omega} |\nabla v^p|^2 dx \right)^{\frac{3}{4}} \right\} \\ &\leq -2(2p-1)p^{-1} \int_{\Omega} |\nabla u^p|^2 dx - 2(2p-1)p^{-1} \int_{\Omega} |\nabla v^p|^2 dx \\ &\quad + \frac{2pk_1}{3^{\frac{3}{4}}} 2^{\frac{1}{2}} \left\{ \left(\frac{3}{2\rho} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{2p} dx \right)^{\frac{3}{2}} \right. \\ &\quad \quad \left. + \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \left[\frac{\beta_1^{-3}}{4} \left(\int_{\Omega} u^{2p} dx \right)^3 + \frac{3\beta_1}{4} \left(\int_{\Omega} |\nabla u^p|^2 dx \right)^{\frac{3}{4}} \right] \right\} \\ &\quad + \frac{2pk_2}{3^{\frac{3}{4}}} 2^{\frac{1}{2}} \left\{ \left(\frac{3}{2\rho} \right)^{\frac{3}{2}} \left(\int_{\Omega} v^{2p} dx \right)^{\frac{3}{2}} \right. \\ &\quad \quad \left. + \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \left[\frac{\beta_2^{-3}}{4} \left(\int_{\Omega} v^{2p} dx \right)^3 + \frac{3\beta_2}{4} \left(\int_{\Omega} |\nabla v^p|^2 dx \right)^{\frac{3}{4}} \right] \right\},\end{aligned}\tag{3.6}$$

where β_1 and β_2 are positive constants satisfying

$$-2(2p-1)p^{-1} + \frac{3^{\frac{1}{4}}pk_1}{2^{\frac{1}{2}}} \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \beta_1 \leq 0, \quad -2(2p-1)p^{-1} + \frac{3^{\frac{1}{4}}pk_2}{2^{\frac{1}{2}}} \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \beta_1 \leq 0.\tag{3.7}$$

Accordingly,

$$\begin{aligned}
\mathcal{E}'(t) &\leq \frac{2p k_1}{3^{\frac{3}{4}}} 2^{\frac{1}{2}} \left\{ \left(\frac{3}{2\rho} \right)^{\frac{3}{2}} \left(\int_{\Omega} u^{2p} dx \right)^{\frac{3}{2}} + \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \left[\frac{\beta_1^{-3}}{4} \left(\int_{\Omega} u^{2p} dx \right)^3 \right] \right\} + \\
&\quad \frac{2p k_2}{3^{\frac{3}{4}}} 2^{\frac{1}{2}} \left\{ \left(\frac{3}{2\rho} \right)^{\frac{3}{2}} \left(\int_{\Omega} v^{2p} dx \right)^{\frac{3}{2}} + \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \left[\frac{\beta_2^{-3}}{4} \left(\int_{\Omega} v^{2p} dx \right)^3 \right] \right\}, \quad (3.8) \\
&\leq 3^{\frac{3}{4}} p k \rho^{-\frac{3}{2}} \left(\int_{\Omega} (u^{2p} + v^{2p}) dx \right)^{\frac{3}{2}} + \frac{p k}{2^{\frac{1}{2}} 3^{\frac{3}{4}}} \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \beta^{-3} \left(\int_{\Omega} u^{2p} + v^{2p} dx \right)^3, \quad (3.9)
\end{aligned}$$

where $k = \max(k_1, k_2)$ and $\beta = \min(\beta_1, \beta_2)$. As a result, we obtain

$$\mathcal{E}'(t) \leq K_1 \mathcal{E}^{\frac{3}{2}}(t) + K_2 \mathcal{E}^3(t), \quad (3.10)$$

where

$$K_1 = 3^{\frac{3}{4}} p k \rho^{-\frac{3}{2}}, \quad K_2 = \frac{p k}{2^{\frac{1}{2}} 3^{\frac{3}{4}}} \left(\frac{d}{\rho} + 1 \right)^{\frac{3}{2}} \beta^{-3}. \quad (3.11)$$

Integrating yields

$$t^* \geq \int_{\mathcal{E}(0)}^{\infty} \frac{d\xi}{K_1 \xi^{\frac{3}{2}} + K_2 \xi^3}. \quad (3.12)$$

This completes the proof of the theorem. \square

We remark that Theorem 3.1 remains true if the Robin boundary conditions are replaced by Neumann boundary conditions. From the proof of Theorem 3.1, it is readily seen that the following result is true under an assumption which is weaker than (A2) and (A3).

Theorem 3.2. *Let $(u(x, t), v(x, t))$ be a pair of nonnegative solution of (1.1) and that at least one of $u(x, t)$ and $v(x, t)$ blow up in finite time, $t = t^*$. Suppose that $(A1)' \sim (A2)'$ below are satisfied:*

(A1)' $\Omega \in \mathbb{R}^3$ is a bounded smooth convex domain;

(A2)' For $k_1, k_2, u, v > 0$ and $p \geq 1$,

$$u^{2p-1} f_1(u, v) + v^{2p-1} f_2(u, v) \leq k_1 u^{3p} + k_2 v^{3p}. \quad (3.13)$$

Then t^ is bounded below by (3.12) .*

Now we are in the position to apply the following results to give an example of the above theorem.

Theorem 3.3 (From [2]). *Consider the following reaction-diffusion system with absorption:*

$$\begin{cases} u_t = \Delta u + v^p - a u^r, & \text{in } \Omega \times (0, \infty), \\ v_t = \Delta v + u^q - b v^s, & \text{in } \Omega \times (0, \infty), \\ u = 0, \quad v = 0, & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = g_1(x), \quad v(x, 0) = g_2(x) & \text{in } \Omega, \end{cases} \quad (3.14)$$

where $\Omega \in \mathbb{R}^n$; a, b, p, q, r and s are positive constants; g_1 and g_2 are nonnegative functions.

- (i) *If $p q > \max(r, 1) \max(s, 1)$, then there exist solutions of (3.14) which blow up in finite time.*
- (ii) *If $p q < \max(r, 1) \max(s, 1)$, then all solutions of (3.14) are global. Moreover, if $r, s \geq 1$ (hence $p q < r s$), they are uniformly bounded.*
- (iii) *If $p q = \max(r, 1) \max(s, 1)$, then*
 - (a) *if $r, s > 1$ and a and b are sufficiently small, then there exist solutions of (3.14) which blow up in finite time;*
 - (b) *if $r, s \geq 1$, and $a^q b^r \geq 1$ (equivalently, $a^s b^p \geq 1$), then all solutions are global and uniformly bounded;*
 - (c) *if r or $s \leq 1$, then all solutions are global (possibly bounded).*

For the nonlinearity $f_1 = v^3 - a u^3$ and $f_2 = u^3 - b v^3$ (by choosing $p = q = r = s = 3$ in Theorem 3.3), we let $p = 2$ in Theorem 3.2 so that

$$u^{2p-1} f_1(u, v) + v^{2p-1} f_2(u, v) = 2 u^3 v^3 - a u^6 - b v^6, \quad (3.15)$$

while

$$k_1 u^{3p} + k_2 v^{3p} = k_1 u^6 + k_2 v^6. \quad (3.16)$$

By choosing $k_1, k_2 \geq 1$, it is easy to see that (3.13) automatically holds for any $a, b > 0$. We apply Theorem 3.3 (iii) – (a) to conclude that (3.14) with $p = q = r = s = 3$ and a, b sufficiently small admits a blow-up solution. Now Theorem 3.2 provides a lower bounded for blow-up time $t = t^*$.

4 Concluding Remarks

In Theorem 1.1 and Theorem 3.1, the estimates of the blow-up time for the blow-up solutions of (1.1) are given by means of the differential inequality technique adapted from [10]. In the present paper, we have shown that this technique can also be applied to certain reaction-diffusion *systems*. From the proofs of Theorem 1.1 and Theorem 3.1

however, we cannot determine whether one of u and v blows up or if both u and v blow up. This problem is left for future work.

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